

Combinatorics and topology of toric arrangements defined by root systems

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Abstract

Given the toric (or toral) arrangement defined by a root system Φ , we describe the poset of its *layers* (connected components of intersections) and we count its elements. Indeed we show how to reduce to 0-dimensional layers, and in this case we provide an explicit formula involving the maximal subdiagrams of the affine Dynkin diagram of Φ . Then we compute the Euler characteristic and the Poincaré polynomial of the complement of the arrangement, which is the set of regular points of the torus.

1 Introduction

Let \mathfrak{g} be a semisimple Lie algebra of rank n over \mathbb{C} , \mathfrak{h} a Cartan subalgebra, $\Phi \subset \mathfrak{h}^*$ and $\Phi^\vee \subset \mathfrak{h}$ respectively the root and coroot systems. The equations $\{\alpha(h) = 0, \alpha \in \Phi\}$ define in \mathfrak{h} a family \mathcal{H} of intersecting hyperplanes. Let $\langle \Phi^\vee \rangle$ be the lattice spanned by the coroots: the quotient $T \doteq \mathfrak{h} / \langle \Phi^\vee \rangle$ is a complex torus of rank n . Each root α takes integer values on $\langle \Phi^\vee \rangle$, hence it induces a map $T \rightarrow \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote by e^α . This is a character of T ; let H_α be its kernel:

$$H_\alpha \doteq \{t \in T \mid e^\alpha(t) = 1\}.$$

In this way Φ defines in T a finite family of hypersurfaces

$$\mathcal{T} \doteq \{H_\alpha, \alpha \in \Phi^+\}$$

(since clearly $H_\alpha = H_{-\alpha}$). \mathcal{H} and \mathcal{T} are called respectively the *hyperplane arrangement* and the *toric arrangement* defined by Φ (see for instance [8], [10], [23]). We call *spaces* of \mathcal{H} the intersections of elements of \mathcal{H} , and *layers* of \mathcal{T} the connected components of the intersections of elements of \mathcal{T} . We denote by $\mathcal{L}(\Phi)$ the set of the spaces of \mathcal{H} , by $\mathcal{C}(\Phi)$ the set of the layers of \mathcal{T} , and by $\mathcal{L}_d(\Phi)$ and $\mathcal{C}_d(\Phi)$ the sets of d -dimensional spaces and layers. Clearly

if $\Phi = \Phi_1 \times \Phi_2$ then $\mathcal{L}(\Phi) = \mathcal{L}(\Phi_1) \times \mathcal{L}(\Phi_2)$ and $\mathcal{C}(\Phi) = \mathcal{C}(\Phi_1) \times \mathcal{C}(\Phi_2)$, hence from now on we will suppose Φ to be irreducible. Let W be the Weyl group of Φ : since W permutes the roots, its natural action on T restricts to an action on $\mathcal{C}(\Phi)$.

\mathcal{H} is a classical object, whereas \mathcal{T} has recently been shown ([8]) to provide a geometric way to compute the values of the Kostant partition function. This function counts in how many ways an element of the lattice $\langle \Phi \rangle$ can be written as sum of positive roots, and plays an important role in representation theory, since (by Kostant's and Steinberg's formulae [19], [26]) it yields efficient computation of weight multiplicities and Littlewood-Richardson coefficients, as shown in [6] using results from [1], [3], [7], [27]. The values of Kostant partition function can be computed as a sum of contributions given by the elements of $\mathcal{C}_0(\Phi)$ (see [6, Teor 3.2]).

Furthermore, let \mathcal{R}_Φ be the complement in T of the union of all elements of \mathcal{T} . \mathcal{R}_Φ is known as the set of the *regular points* of the torus T and has been widely studied (see in particular [8], [20], [21]). The cohomology of \mathcal{R}_Φ is direct sum of contributions given by the elements of $\mathcal{C}(\Phi)$ (see for instance [8]). Then by describing the action of W on $\mathcal{C}(\Phi)$ we implicitly obtain a W -equivariant decomposition of the cohomology of \mathcal{R}_Φ , and by counting and classifying the elements of $\mathcal{C}(\Phi)$ we can compute the Poincaré polynomial of \mathcal{R}_Φ .

We say that a subset Θ of Φ is a *subsystem* if it satisfies the following conditions:

1. $\alpha \in \Theta \Rightarrow -\alpha \in \Theta$
2. $\alpha, \beta \in \Theta$ and $\alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Theta$.

For each $t \in T$ let us define the following subsystem of Φ :

$$\Phi(t) \doteq \{\alpha \in \Phi | e^\alpha(t) = 1\}.$$

and denote by $W(t)$ the stabilizer of t .

The aim of Section 2 is to describe $\mathcal{C}_0(\Phi)$, which is the set of points $t \in T$ such that $\Phi(t)$ has rank n . We call its elements the *points* of the arrangement \mathcal{T} . Let $\alpha_1, \dots, \alpha_n$ be simple roots of Φ , α_0 the lowest root (i.e. the opposite of the highest root), and Φ_p the subsystem of Φ generated by $\{\alpha_i\}_{0 \leq i \leq n, i \neq p}$. Let Γ be the affine Dynkin diagram of Φ and $V(\Gamma)$ the set of its vertices (a list of such diagrams can be found for instance in [13] or in [18]). $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$, hence we can identify each vertex p with an integer from 0 to n . The diagram Γ_p obtained by removing from Γ the vertex p (and all adjacent edges) is the ordinary Dynkin diagram of Φ_p . Let W_p be the Weyl group of Φ_p , i.e. the subgroup of W generated

by all the reflections $s_{\alpha_0}, \dots, s_{\alpha_n}$ except s_{α_p} . Notice that Γ_0 is the Dynkin diagram of Φ and $W_0 = W$.

Then we prove:

Theorem 1. *There is a bijection between the W -orbits of $\mathcal{C}_0(\Phi)$ and the vertices of Γ , having the property that for every point t in the orbit \mathcal{O}_p corresponding to the vertex p , $\Phi(t)$ is W -conjugate to Φ_p and $W(t)$ is W -conjugate to W_p .*

As a corollary we get the formula

$$|\mathcal{C}_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}. \quad (1)$$

In Section 3 we deal with layers of arbitrary dimension. For each layer C of \mathcal{T} we consider the subsystem of Φ

$$\Phi_C \doteq \{\alpha \in \Phi | e^\alpha(t) = 1 \ \forall t \in C\}$$

and its *completion* $\overline{\Phi_C} \doteq \langle \Phi_C \rangle_{\mathbb{R}} \cap \Phi$.

Let \mathcal{K}_d be the set of subsystems Θ of Φ of rank $n - d$ that are *complete* (i.e. such that $\Theta = \overline{\Theta}$), and let \mathcal{C}_Θ^Φ be the set of layers C such that $\overline{\Phi_C} = \Theta$. This gives a partition of the layers:

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}_\Theta^\Phi.$$

Notice that the subsystem of roots vanishing on a space of \mathcal{H} is always complete; then \mathcal{K}_d is in bijection with \mathcal{L}_d . The elements of \mathcal{L}_d are classified and counted in [22], [23]. Thus the description of the sets \mathcal{C}_Θ^Φ given in Theorem 11 yields a classification of the layers of \mathcal{T} . In particular we show that $|\mathcal{C}_\Theta^\Phi| = n_\Theta^{-1} |\mathcal{C}_0(\Theta)|$, where n_Θ is a natural number depending only on the conjugacy class of Θ , and then

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |\mathcal{C}_0(\Theta)|.$$

In Section 4, using results of [8] and [9], we deduce from Theorem 1 that the Euler characteristic of \mathcal{R}_Φ is equal to $(-1)^n |W|$. Moreover, Corollary 12 yields a formula for the Poincaré polynomial of \mathcal{R}_Φ :

$$P_\Phi(q) = \sum_{d=0}^n (-1)^d (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |W^\Theta|.$$

By this formula $P_\Phi(q)$ can be explicitly computed.

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2 Points of the arrangement

2.1 Statements

For all facts about Lie algebras and root systems we refer to [15]. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the Cartan decomposition of \mathfrak{g} , and let us choose nonzero elements

$$X_0, X_1, \dots, X_n$$

in the one-dimensional subalgebras $\mathfrak{g}_{\alpha_0}, \mathfrak{g}_{\alpha_1}, \dots, \mathfrak{g}_{\alpha_n}$: since $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] = \mathfrak{g}_{\alpha+\alpha'}$ whenever $\alpha, \alpha', \alpha + \alpha' \in \Phi$, we have that X_0, X_1, \dots, X_n generate \mathfrak{g} . Let $a_0 = 1$ and for $p = 1, \dots, n$ let a_p be the coefficient of α_p in $-\alpha_0$. For each $p = 0, \dots, n$ we define an automorphism σ_p of \mathfrak{g} by

$$\sigma_p(X_j) \doteq \begin{cases} X_j & \text{if } j \neq p \\ e^{2\pi i a_p^{-1}} X_j & \text{if } j = p \end{cases}$$

Let G be the semisimple and simply connected linear algebraic group having root system Φ ; then \mathfrak{g} is the Lie algebra of G , and T is the maximal torus of G corresponding to \mathfrak{h} (see for instance [14]). G acts on itself by conjugacy, and for each $g \in G$ the map $k \mapsto gkg^{-1}$ is an automorphism of G . Its differential $Ad(g)$ is an automorphism of \mathfrak{g} .

Remark 2. For every $t \in \mathcal{C}_0(\Phi)$, let $\mathfrak{g}^{Ad(t)}$ be the subalgebra of the elements fixed by $Ad(t)$. For every $\alpha \in \Phi$ and for every $X_\alpha \in \mathfrak{g}_\alpha$ we have that

$$Ad(t)(X_\alpha) = e^\alpha(t) X_\alpha$$

and then

$$\mathfrak{g}^{Ad(t)} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(t)} \mathfrak{g}_\alpha.$$

On the other hand \mathfrak{g}^{σ_p} is generated by the subalgebras $\{\mathfrak{g}_{\alpha_i}\}_{0 \leq i \leq n, i \neq p}$. Then $\mathfrak{g}^{Ad(t)}$ and \mathfrak{g}^{σ_p} are semisimple algebras having root system respectively $\Phi(t)$ and Φ_p . Our strategy will be to prove that for each $t \in \mathcal{C}_0(\Phi)$, $Ad(t)$ is conjugate to some σ_p . This implies that $\mathfrak{g}^{Ad(t)}$ is conjugate to \mathfrak{g}^{σ_p} and then $\Phi(t)$ to Φ_p , as claimed in Theorem 1.

Then we want to give a bijection between vertices of Γ and W -orbits of $\mathcal{C}_0(\Phi)$ showing that, for every t in the orbit \mathcal{O}_p , $Ad(t)$ is conjugate to σ_p . However, since some of the σ_p (as well as the corresponding Φ_p) are themselves conjugate, this bijection is not canonical. To make it canonical we should merge the orbits corresponding to conjugate automorphisms: for this we consider the action of a larger group.

Let $\Lambda(\Phi) \subset \mathfrak{h}$ be the lattice of the *coweights* of Φ , i.e.

$$\Lambda(\Phi) \doteq \{h \in \mathfrak{h} | \alpha(h) \in \mathbb{Z} \ \forall \alpha \in \Phi\}.$$

The lattice spanned by the coroots $\langle \Phi^\vee \rangle$ is a sublattice of $\Lambda(\Phi)$; set

$$Z(\Phi) \doteq \frac{\Lambda(\Phi)}{\langle \Phi^\vee \rangle}.$$

This finite subgroup of T coincides with $Z(G)$, the *center* of G . It is well known (see for instance [14, 13.4]) that

$$Ad(g) = id_{\mathfrak{g}} \Leftrightarrow g \in Z(\Phi). \quad (2)$$

Notice that

$$Z(\Phi) = \{t \in T | \Phi(t) = \Phi\}$$

thus $Z(\Phi) \subseteq \mathcal{C}_0(\Phi)$. Moreover, for each $z \in Z(\Phi), t \in T, \alpha \in \Phi$,

$$e^\alpha(zt) = e^\alpha(z)e^\alpha(t) = e^\alpha(t)$$

and therefore $\Phi(zt) = \Phi(t)$. In particular $Z(\Phi)$ acts by multiplication on $\mathcal{C}_0(\Phi)$. Notice that this action commutes with that of W : indeed, let

$$N \doteq N_G(T)$$

be the normalizer of T in G . We recall that $W \simeq N/T$ and the action of W on T is induced by the conjugacy action of N . The elements of $Z(\Phi) = Z(G)$ commute with the elements of G , hence in particular with the elements of N . Thus we get an action of $W \times Z(\Phi)$ on $\mathcal{C}_0(\Phi)$.

Let Q be the set of the $Aut(\Gamma)$ -orbits of $V(\Gamma)$. If $p, p' \in V(\Gamma)$ are two representatives of $q \in Q$, then $\Gamma_p \simeq \Gamma_{p'}$, thus $W_p \simeq W_{p'}$. Moreover we will see (Corollary 7(ii)) that σ_p is conjugate to $\sigma_{p'}$. Then we can restate Theorem 1 as follows.

Theorem 3. *There is a canonical bijection between Q and the set of $W \times Z(\Phi)$ -orbits in $\mathcal{C}_0(\Phi)$, having the property that if $p \in V(\Gamma)$ is a representative of $q \in Q$, then:*

1. *every point t in the corresponding orbit \mathcal{O}_q induces an automorphism conjugate to σ_p ;*
2. *the stabilizer of $t \in \mathcal{O}_q$ is isomorphic to $W_p \times Stab_{Aut(\Gamma)}p$.*

This theorem implies immediately the formula:

$$|\mathcal{C}_0(\Phi)| = \sum_{q \in Q} |q| \frac{|W|}{|W_p|} \quad (3)$$

where p is any representative of q . This is clearly equivalent to formula (1).

Remark 4. If we view the elements of $\Lambda(\Phi)$ as translations, we can define a group of isometries of \mathfrak{h}

$$\widetilde{W} \doteq W \ltimes \Lambda(\Phi).$$

\widetilde{W} is called the *extended affine Weyl group* of Φ and contains the affine Weyl group $\widehat{W} \doteq W \ltimes \langle \Phi^\vee \rangle$ (see for instance [16], [24]).

The action of $W \times Z(\Phi)$ on $\mathcal{C}_0(\Phi)$ is induced by that of \widetilde{W} . Indeed \widetilde{W} preserves the lattice $\langle \Phi^\vee \rangle$ of \mathfrak{h} , and thus acts on $T = \mathfrak{h}/\langle \Phi^\vee \rangle$ and on $\mathcal{C}_0(\Phi) \subset T$. Since the semidirect factor $\langle \Phi^\vee \rangle$ acts trivially, \widetilde{W} acts as its quotient

$$\frac{\widetilde{W}}{\langle \Phi^\vee \rangle} \simeq W \times Z(\Phi).$$

2.2 Examples: the classical root systems

In the following examples we denote by \mathfrak{S}_n , \mathfrak{D}_n , \mathfrak{C}_n respectively the symmetric, dihedral and cyclic group on n letters.

1. Case \mathfrak{C}_n The roots

$$2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n$$

($i = 1, \dots, n$) take integer values on the points $[\alpha_1^\vee/2], \dots, [\alpha_n^\vee/2] \in \mathfrak{h}/\langle \Phi^\vee \rangle$, and thus on their sums, for a total of 2^n points of $\mathcal{C}_0(\Phi)$. Indeed, let us introduce the following notation. Fixed a basis h_1^*, \dots, h_n^* of \mathfrak{h}^* , the simple roots of \mathfrak{C}_n can be written as

$$\alpha_i = h_i^* - h_{i+1}^* \text{ for } i = 1, \dots, n-1, \text{ and } \alpha_n = 2h_n^*. \quad (4)$$

Then

$$\Phi = \{h_i^* - h_j^*\} \cup \{h_i^* + h_j^*\} \cup \{\pm 2h_i^*\} \text{ (} i, j = 1, \dots, n, i \neq j \text{)}$$

and writing t_i for $e^{h_i^*}$, we have that

$$e^\Phi \doteq \{e^\alpha, \alpha \in \Phi\} = \{t_i t_j^{-1}\} \cup \{t_i t_j\} \cup \{t_i^{\pm 2}\}.$$

The system of n independent equations

$$\begin{cases} t_1^2 = 1 \\ \dots \\ t_n^2 = 1 \end{cases}$$

has 2^n solutions: $(\pm 1, \dots, \pm 1)$, and it is easy to see that all other systems does not have other solutions. The Weyl group $W \simeq \mathfrak{S}_n \ltimes (\mathfrak{C}_2)^n$ acts on $T = (\mathbb{C}^*)^n$ by permuting and inverting its coordinates; the second operation is trivial on $\mathcal{C}_0(\Phi)$. Thus two elements of $\mathcal{C}_0(\Phi)$ are in the same W -orbit if and only if they have the same number of negative coordinates. Then we can define the p -th W -orbit \mathcal{O}_p as the set of points with p negative coordinates. (This choice is not canonical: we may choose the set of points with p positive coordinates as well). Clearly if $t \in \mathcal{O}_p$ then

$$W(t) \simeq (\mathfrak{S}_p \times \mathfrak{S}_{n-p}) \ltimes (\mathfrak{C}_2)^n.$$

Thus $|\mathcal{O}_p| = \binom{n}{p}$ and we get:

$$|\mathcal{C}_0(\Phi)| = \sum_{p=0}^n \binom{n}{p} = 2^n.$$

Notice that if $t \in \mathcal{O}_p$ then $-t \in \mathcal{O}_{n-p}$, and $Ad(t) = Ad(-t)$ since $Z(\Phi) = \{\pm(1, \dots, 1)\}$. In fact Γ has a symmetry exchanging the vertices p and $n-p$. Finally notice that $\mathcal{C}_0(\Phi)$ is a subgroup of T isomorphic to $(\mathfrak{C}_2)^n$ and generated by the elements

$$\delta_i \doteq (1, \dots, 1, -1, 1, \dots, 1) \text{ (with the } -1 \text{ at the } i\text{-th place)}.$$

Then we can come back to the original coordinates observing that δ_i is the nontrivial solution of the system $t_i^2 = 1$, $t_j = 1 \forall j \neq i$, and using (6) to get:

$$\delta_i \leftrightarrow \left[\sum_{k=i}^n \alpha_k^\vee / 2 \right].$$

2. **Case D_n** We can write $\alpha_n = h_{n-1}^* + h_n^*$ and the others α_i as before; then

$$e^\Phi = \{t_i t_j^{-1}\} \cup \{t_i t_j\}.$$

Then each system of n independent equations is W -conjugate to one of this form:

$$\begin{cases} t_1 = t_2 \\ \dots \\ t_{p-1} = t_p \\ t_{p-1} = t_p^{-1} \\ t_{p+1}^{\pm 1} = t_{p+2} \\ \dots \\ t_{n-1} = t_n \\ t_{n-1} = t_n^{-1} \end{cases}$$

for some $p \neq 1, n-1$. Then we get the subset of $(\mathfrak{C}_2)^n$ composed by the following n -ples:

$$\{(\pm 1, \dots, \pm 1)\} \setminus \{\pm \delta_i, i = 1, \dots, n\}$$

which are in number of $2^n - 2n$. However reasoning as before we see that each one represents two points in $\mathfrak{h}/\langle \Phi^\vee \rangle$. Namely, the correspondence is given by:

$$\left\{ \left[\sum_{k=i}^{n-1} \frac{\alpha_k^\vee}{2} \pm \frac{\alpha_{n-1}^\vee - \alpha_n^\vee}{4} \right] \right\} \longrightarrow \delta_i.$$

From a geometric point of view, the t_i s are coordinates of a maximal torus of the orthogonal group, while $T = \mathfrak{h}/\langle \Phi^\vee \rangle$ is a maximal torus of its two-sheets universal covering. Each W -orbit corresponding to the four extremal vertices of Γ is a singleton consisting of one of the four points over $\pm(1, \dots, 1)$, all inducing the identity automorphism: indeed $Aut(\Gamma)$ acts transitively on these points. The other orbits are defined as in the case \mathcal{C}_n .

3. **Case \mathcal{B}_n** This case is very similar to the previous one, but now $\alpha_n = h_n^*$,

$$e^\Phi = \{t_i t_j^{-1}\} \cup \{t_i t_j\} \cup \{t_i^{\pm 1}\}$$

and then we get the points

$$\{(\pm 1, \dots, \pm 1)\} \setminus \{\delta_i\}_{i=1, \dots, n}.$$

In this case the projection is

$$\left\{ \left[\sum_{k=i}^{n-1} \frac{\alpha_k^\vee}{2} \pm \frac{\alpha_n^\vee}{4} \right] \right\} \longrightarrow \delta_i$$

then we have $2^n - n$ pairs of points in $\mathcal{C}_0(\Phi)$.

4. **Case \mathcal{A}_n** If we see \mathfrak{h}^* as the subspace of $\langle h_1^*, \dots, h_{n+1}^* \rangle$ of equation $\sum h_i^* = 0$, and T as the subgroup of $(\mathbb{C}^*)^{n+1}$ of equation $\prod t_i = 1$, we can write all the simple roots as $\alpha_i = h_i^* - h_{i+1}^*$; then $e^\Phi = \{t_i t_j^{-1}\}$. In this case Φ has no proper subsystem of its same rank, then all the coordinates must be equal. Therefore

$$\mathcal{C}_0(\Phi) = Z(\Phi) = \{(\zeta, \dots, \zeta) | \zeta^{n+1} = 1\} \simeq \mathfrak{C}_{n+1}.$$

Then $W \simeq \mathfrak{S}_{n+1}$ acts on $\mathcal{C}_0(\Phi)$ trivially and $Z(\Phi)$ transitively, as expected since $Aut(\Gamma) \simeq \mathfrak{D}_{n+1}$ acts transitively on the vertices of Γ . We can write more explicitly $\mathcal{C}_0(\Phi) \subseteq \mathfrak{h}/\langle \Phi^\vee \rangle$ as

$$\mathcal{C}_0(\Phi) = \left\{ \left[\frac{k}{n+1} \sum_{i=1}^n i \alpha_i^\vee \right], k = 0, \dots, n \right\}.$$

2.3 Proofs

Motivated by Remark 2, we start to describe the automorphisms of \mathfrak{g} that are induced by the points of $\mathcal{C}_0(\Phi)$.

Lemma 5. *If $t \in \mathcal{C}_0(\Phi)$, then $Ad(t)$ has finite order.*

Proof. Let β_1, \dots, β_n linearly independent roots such that $e^{\beta_i}(t) = 1$: then for each root $\alpha \in \Phi$ we have that $m\alpha = \sum c_i \beta_i$ for some m and $c_i \in \mathbb{Z}$, and thus

$$e^\alpha(t^m) = e^{m\alpha}(t) = \prod_{i=1}^n (e^{\beta_i})^{c_i}(t) = 1.$$

Then $Ad(t^m)$ is the identity on \mathfrak{g} , hence by (2) $t^m \in Z(\Phi)$. $Z(\Phi)$ is a finite group, thus t^m and t have finite order. \square

The previous lemma allows us to apply the following

Theorem 6 (Kač).

1. *Each inner automorphism of \mathfrak{g} of finite order m is conjugate to an automorphism σ of the form*

$$\sigma(X_i) = \zeta^{s_i} X_i$$

with ζ fixed primitive m -th root of unity and (s_0, \dots, s_n) nonnegative integers without common factors such that $m = \sum s_i a_i$.

2. *Two such automorphisms are conjugate if and only if there is an automorphism of Γ sending the parameters (s_0, \dots, s_n) of the first in the parameters (s'_0, \dots, s'_n) of the second.*
3. *Let (i_1, \dots, i_r) be all the indices for which $s_{i_1} = \dots = s_{i_r} = 0$. Then \mathfrak{g}^σ is the direct sum of an $(n-r)$ -dimensional center and of a semisimple Lie algebra whose Dynkin diagram is the subdiagram of Γ of vertices i_1, \dots, i_r .*

This is a special case of a theorem proved in [17] and more extensively in [13, X.5.15 and 16]. We only need the following

Corollary 7.

1. *Let σ be an inner automorphism of \mathfrak{g} of finite order m such that \mathfrak{g}^σ is semisimple. Then there is $p \in V(\Gamma)$ such that σ is conjugate to σ_p . In particular $m = a_p$ and the Dynkin diagram of \mathfrak{g}^σ is Γ_p .*
2. *Two automorphisms $\sigma_p, \sigma_{p'}$ are conjugate if and only if p, p' are in the same $Aut(\Gamma)$ -orbit.*

Proof. If \mathfrak{g}^σ is semisimple, then in the third part of Theorem 6 $n = r$, hence all parameters of σ but one are equal to 0, and the nonzero parameter s_p must be equal to 1, otherwise there would be a common factor, contradicting the first part of the Theorem. Thus we get the first statement. Then the second statement follows from Theorem 6(ii). \square

Let be $t \in \mathcal{C}_0(\Phi)$: by Remark 2 $\mathfrak{g}^{Ad(t)}$ is semisimple, hence by Corollary 7(i) $Ad(t)$ is conjugate to some σ_p . Then there is a canonical map

$$\begin{aligned} \psi : \mathcal{C}_0(\Phi) &\rightarrow Q \\ t &\mapsto \psi(t) = \{p \in V(\Gamma) \text{ such that } \sigma_p \text{ is conjugate to } Ad(t)\}. \end{aligned}$$

Notice that $\psi(t)$ is a well-defined element of Q by Corollary 7(ii).

We now prove the fundamental

Lemma 8. *Two points in $\mathcal{C}_0(\Phi)$ induce conjugate automorphisms if and only if they are in the same $W \times Z(\Phi)$ -orbit.*

Proof. We recall that $W \simeq N/T$ and the action of W on T is induced by the conjugation action of N ; it is also well known that two points of T are G -conjugate if and only if they are W -conjugate. Then W -conjugate points induce conjugate automorphisms. Moreover by (2)

$$Ad(t) = Ad(s) \Leftrightarrow Ad(ts^{-1}) = id_{\mathfrak{g}} \Leftrightarrow ts^{-1} \in Z(\Phi).$$

Finally suppose that $t, t' \in \mathcal{C}_0(\Phi)$ induce conjugate automorphisms, i.e.

$$\exists g \in G | Ad(t') = Ad(g)Ad(t)Ad(g^{-1}) = Ad(gt g^{-1}).$$

Then $zt' = gt g^{-1}$ for some $z \in Z(\Phi)$. Thus zt' and t are G -conjugate elements of T , and hence they are W -conjugate, proving the claim. \square

We can now prove the first part of Theorem 3. Indeed by the previous lemma there is a canonical injective map defined on the set of the orbits of $\mathcal{C}_0(\Phi)$:

$$\overline{\psi} : \frac{\mathcal{C}_0(\Phi)}{W \times Z(\Phi)} \longrightarrow Q.$$

We must show that this map is surjective. The system

$$\alpha_i(h) = 1 \ (\forall i \neq 0, p), \ \alpha_p(h) = a_p^{-1}$$

is composed of n linearly independent equations, then it has a solution $h \in \mathfrak{h}$. Notice that $\alpha_0(h) \in \mathbb{Z}$. Let t be the class of h in T ; then

$$e^\alpha(t) = 1 \Leftrightarrow \alpha \in \Phi_p.$$

Then by Remark 2 $Ad(t)$ is conjugate to σ_p and $\Phi(t)$ to Φ_p .

In order to relate the action of $Z(\Phi)$ with that of $Aut(\Gamma)$, we introduce the following subset of W . For each $p \neq 0$ such that $a_p = 1$, set $z_p \doteq w_0^p w_0$, where w_0 is the longest element of W and w_0^p is the longest element of the parabolic subgroup of W generated by all the simple reflections $s_{\alpha_1}, \dots, s_{\alpha_n}$ except s_{α_p} . Then we define

$$W_Z \doteq \{1\} \cup \{z_p\}_{p=1, \dots, n | a_p=1}$$

W_Z has the following properties (see [16, 1.7 and 1.8]):

Theorem 9 (Iwahori-Matsumoto).

1. W_Z is a subgroup of W isomorphic to $Z(\Phi)$.
2. For each $z_p \in W_Z$, we have that $z_p \cdot \alpha_0 = \alpha_p$, and z_p induces an automorphism of Γ that sends the 0-th vertex to the p -th one; this defines an injective morphism $W_Z \hookrightarrow Aut(\Gamma)$.
3. The W_Z -orbits of $V(\Gamma)$ coincide with the $Aut(\Gamma)$ -orbits.

Therefore Q is the set of W_Z -orbits of $V(\Gamma)$, and the bijection $\overline{\psi}$ between Q and the set of $Z(\Phi)$ -orbits of $\mathcal{C}_0(\Phi)/W$ can be lifted to a noncanonical bijection between $V(\Gamma)$ and $\mathcal{C}_0(\Phi)/W$. Then we just have to consider the action of W on $\mathcal{C}_0(\Phi)$ and prove the

Lemma 10. *If $t \in \mathcal{O}_p$, then $W(t)$ is conjugate to W_p .*

Proof. Notice that the centralizer $C_N(t)$ of t in N is the normalizer of $T = C_T(t)$ in $C_G(t)$. Then $W(t) = C_N(t)/T$ is the Weyl group of $C_G(t)$. $C_G(t)$ is the subgroup of G of points fixed by the conjugacy by t , then its Lie algebra is $\mathfrak{g}^{Ad(t)}$, which is conjugate to \mathfrak{g}^{σ_p} by the first part of Theorem 3. Therefore $W(t)$ is conjugate to W_p . □

This completes the proof of Theorem 3 and also of Theorem 1, since by Remark 2 the map ψ defined in (7) can also be seen as the map

$$t \mapsto \psi(t) = \{p \in V(\Gamma) \text{ such that } \Phi_p \text{ is conjugate to } \Phi(t)\}.$$

3 Layers of the arrangement

3.1 From hyperplane arrangements to toric arrangements

Let S be a d -dimensional space of \mathcal{H} . The set Φ_S of the elements of Φ vanishing on S is a complete subsystem of Φ of rank $n - d$. Then the map $S \rightarrow \Phi_S$ gives a bijection between \mathcal{L}_d and \mathcal{K}_d , whose inverse is

$$\Theta \rightarrow S(\Theta) \doteq \{h \in \mathfrak{h} \mid \alpha(h) = 0 \ \forall \alpha \in \Theta\}.$$

In [23, 6.4 and C] (following [22] and [5]) the spaces of \mathcal{H} are classified and counted, and the W -orbits of \mathcal{L}_d are completely described. This is done case-by-case according to the type of Φ . We now show a case-free way to extend this analysis to the layers of \mathcal{T} .

Given a layer C of \mathcal{T} let us consider

$$\Phi_C \doteq \{\alpha \in \Phi \mid e^\alpha(t) = 1 \ \forall t \in C\}.$$

In contrast with the case of linear arrangements, Φ_C in general is not complete. For each $\Theta \in \mathcal{K}_d$, define \mathcal{C}_Θ^Φ as the set of layers C such that $\overline{\Phi_C} = \Theta$. This is clearly a partition of the set of d -dimensional layers of \mathcal{T} , i.e.

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}_\Theta^\Phi \quad (5)$$

Given any $C \in \mathcal{C}_\Theta^\Phi$, we call $S(\Theta)$ the *tangent space* at the layer U . Then by [23] the problem of classifying the layers of \mathcal{T} reduces to classify the layers of \mathcal{T} having a given tangent space, i.e. the elements of \mathcal{C}_Θ^Φ . In the next section we show that this amounts to classify the points of a smaller toric arrangement, namely that defined by Θ .

3.2 Theorems

Let Θ be a complete subsystem of Φ and W^Θ its Weyl group. Let \mathfrak{k} and K be respectively the semisimple Lie algebra and the semisimple and simply connected algebraic group of root system Θ , \mathfrak{d} a Cartan subalgebra of \mathfrak{k} , $\langle \Theta^\vee \rangle$ and $\Lambda(\Theta)$ the coroot and coweight lattices, $Z(\Theta) \doteq \frac{\Lambda(\Theta)}{\langle \Theta^\vee \rangle}$ the center of K , D the maximal torus of K defined by $\mathfrak{d}/\langle \Theta^\vee \rangle$, \mathcal{D} the toric arrangement defined by Θ on D and $\mathcal{C}_0(\Theta)$ the set of its points.

We also consider the *adjoint group* $K_a \doteq K/Z(\Theta)$ and its maximal torus $D_a \doteq D/Z(\Theta) \simeq \mathfrak{d}/\Lambda(\Theta)$. We recall from [14] that K is the universal covering of K_a , and if D' is an algebraic torus having Lie algebra \mathfrak{d} , then $D' \simeq \mathfrak{d}/L$ for some lattice $\Lambda(\Theta) \supseteq L \supseteq \langle \Theta^\vee \rangle$; then there are natural covering projections $D \twoheadrightarrow D' \twoheadrightarrow D_a$ with kernels respectively $L/\langle \Theta^\vee \rangle$ and $\Lambda(\Theta)/L$. Notice that Θ naturally defines an arrangement on each torus D' , and that

for $D' = D_a$ the set of its 0-dimensional layers is $\mathcal{C}_0(\Theta)/Z(\Theta)$. Given a point t of some D' we set

$$\Theta(t) \doteq \{\alpha \in \Theta | e^\alpha(t) = 1\}.$$

Theorem 11. *There is a W^Θ -equivariant surjective map*

$$\varphi : \mathcal{C}_\Theta^\Phi \rightarrow \mathcal{C}_0(\Theta)/Z(\Theta)$$

such that $\ker \varphi \simeq Z(\Phi) \cap Z(\Theta)$ and $\Phi_C = \Theta(\varphi(C))$.

Proof. Let $S(\Theta)$ be the subspace of \mathfrak{h} defined in the previous section, and H the corresponding subtorus of T . T/H is a torus with Lie algebra $\mathfrak{h}/S(\Theta) \simeq \mathfrak{d}$, then Θ defines an arrangement \mathcal{D}' on $D' \doteq T/H$. The projection $\pi : T \twoheadrightarrow T/H$ induces a bijection between \mathcal{C}_Θ^Φ and the set of 0-dimensional layers of \mathcal{D}' , because $H \in \mathcal{C}_\Theta^\Phi$ and for each $C \in \mathcal{C}_\Theta^\Phi$, $\Phi_C = \Theta(\pi(C))$.

Moreover the restriction of the projection $d\pi : \mathfrak{h} \twoheadrightarrow \mathfrak{h}/S(\Theta)$ to $\langle \Phi^\vee \rangle$ is simply the map that restricts the coroots of Φ to Θ . Set $R^\Phi(\Theta) \doteq d\pi(\langle \Phi^\vee \rangle)$; then $\Lambda(\Theta) \supseteq R^\Phi(\Theta) \supseteq \langle \Theta^\vee \rangle$ and $D' \simeq \mathfrak{d}/R^\Phi(\Theta)$. Denote by p the projection $\Lambda(\Phi) \twoheadrightarrow \frac{\Lambda(\Phi)}{\langle \Phi^\vee \rangle}$ and embed $\Lambda(\Theta)$ in $\Lambda(\Phi)$ in the natural way. Then the kernel of the covering projection of $D' \twoheadrightarrow D_a$ is isomorphic to

$$\frac{\Lambda(\Theta)}{R^\Phi(\Theta)} \simeq p(\Lambda(\Theta)) \simeq Z(\Phi) \cap Z(\Theta).$$

□

We set

$$n_\Theta \doteq \frac{|Z(\Theta)|}{|Z(\Phi) \cap Z(\Theta)|}.$$

The following corollary is straightforward from Theorem 11.

Corollary 12.

$$|\mathcal{C}_\Theta^\Phi| = n_\Theta^{-1} |\mathcal{C}_0(\Theta)|$$

and then by (5),

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |\mathcal{C}_0(\Theta)|.$$

Notice that two layers C, C' of \mathcal{T} are W -conjugate if and only if the two conditions below are satisfied:

1. their tangent spaces are W -conjugate, i.e. $\exists w \in W$ such that $\overline{\Phi_C} = w.\overline{\Phi_{C'}}$;
2. C and $w.C'$ are $W^{\overline{\Phi_C}}$ -conjugate.

Then the action of W on $\mathcal{C}(\Phi)$ is described by the following remark.

Remark 13.

1. By Theorem 11, φ induces a surjective map $\widehat{\varphi}$ from the set of the W^Θ -orbits of \mathcal{C}_Θ^Φ to the set of the $W^\Theta \times Z(\Theta)$ -orbits of $\mathcal{C}_0(\Theta)$, that are described by Theorem 3.
2. In particular if Θ is irreducible, set Γ^Θ its affine Dynkin diagram, Q^Θ the set of the $\text{Aut}(\Gamma)$ -orbits of its vertices, Γ_p^Θ the diagram that we obtain from Γ^Θ removing the vertex p , and Θ_p the associated root system. Then there is a surjective map

$$\widehat{\varphi} : \mathcal{C}_\Theta^\Phi \twoheadrightarrow Q^\Theta$$

such that, if $\widehat{\varphi}(C) = q$ and p is a representative of q , then $\Phi_C \simeq \Theta_p$.

3.3 Examples

Case F_4 . $Z(\Phi) = \{1\}$, thus $n_\Theta = |Z(\Theta)|$. Therefore in this case n_Θ does not depend on the conjugacy class, but only on the isomorphism class of Θ .

We say that a space S of \mathcal{H} (respectively a layer C of \mathcal{T}) is of a given type if the corresponding subsystem Φ_S (respectively Φ_C) is of that type. Then by [23, Tab. C.9] and Corollary 12 there are:

1. one space of type " A_0 ", tangent to one layer of the same type (the whole spaces);
2. 24 spaces of type A_1 , each tangent to one layer of the same type;
3. 72 spaces of type $A_1 \times A_1$, each tangent to one layer of the same type;
4. 32 spaces of type A_2 , each tangent to one layer of the same type;
5. 18 spaces of type B_2 , each tangent to one layer of the same type and one layer of type $A_1 \times A_1$;
6. 12 spaces of type C_3 , each tangent to one layer of the same type and 3 of type $A_2 \times A_1$;
7. 12 spaces of type B_3 , each tangent to one layer of the same type, one of type A_3 and 3 of type $A_1 \times A_1 \times A_1$;
8. 96 spaces of type $A_1 \times A_2$, each tangent to one layer of the same type;
9. one space of type F_4 (the origin), tangent to: one layer of the same type, 12 of type $A_1 \times C_3$, 32 of type $A_2 \times A_2$, 24 of type $A_3 \times A_1$, and 3 of type C_4 .

Case A_{n-1} . It is easily seen that each subsystem Θ of Φ is complete and is a product of irreducible factors $\Theta_1, \dots, \Theta_k$, with Θ_i of type A_{λ_i-1} for some positive integers λ_i such that $\lambda_1 + \dots + \lambda_k = n$ and $n - k$ is the rank of Θ . In other words, as is well known, the W -conjugacy classes of spaces of \mathcal{H} are in bijection with the partitions λ of n , and if a space has dimension d then corresponding partition has length $|\lambda| \doteq k$ equal to $d + 1$. The number of spaces of partition λ is easily seen to be equal to $n!/b_\lambda$, where b_i is the number of λ_j that are equal to i and $b_\lambda \doteq \prod i^{b_i} b_i!$ (see [23, 6.72]). Now let g_λ be the greatest common divisor of $\lambda_1, \dots, \lambda_k$. By Example 4 in Section 2.2 we have that

$$|Z(\Theta)| = \lambda_1 \dots \lambda_k = |\mathcal{C}_0(\Theta)|$$

and $|Z(\Phi) \cap Z(\Theta)| = g_\lambda$. Then by Corollary 12 $|\mathcal{C}_\Theta^\Phi| = g_\lambda$ and

$$|\mathcal{C}_d(\Phi)| = \sum_{|\lambda|=d+1} \frac{n! g_\lambda}{b_\lambda}.$$

This could also be seen directly as follows. We can view T as the subgroup of $(\mathbb{C}^*)^n$ given by the equation $t_1 \dots t_n - 1 = 0$. Then Θ imposes the equations

$$\begin{cases} t_1 = \dots = t_{\lambda_1} \\ \dots \\ t_{\lambda_1 + \dots + \lambda_{k-1} + 1} = \dots = t_n. \end{cases}$$

Thus we have the relation

$$x_1^{\lambda_1} \dots x_k^{\lambda_k} - 1 = 0.$$

If $g_\lambda = 1$ this polynomial is irreducible, because the vector $(\lambda_1, \dots, \lambda_k)$ can be completed to a basis of the lattice \mathbb{Z}^k . If $g_\lambda > 1$ this polynomial has exactly g_λ irreducible factors over \mathbb{C} . Then in every case it defines an affine variety having g_λ irreducible components, which are precisely the elements of \mathcal{C}_Θ^Φ .

4 Topology of the complement

4.1 Theorems

Let \mathcal{R}_Φ be the complement of the toric arrangement:

$$\mathcal{R}_\Phi \doteq T \setminus \bigcup_{\alpha \in \Phi^+} H_\alpha.$$

In this section we prove that the Euler characteristic of \mathcal{R}_Φ , denoted by E_Φ , is equal to $(-1)^n |W|$. This may also be seen as a consequence of [4,

Prop. 5.3]. Furthermore, we give a formula for the Poincaré polynomial of \mathcal{R}_Φ , denoted by $P_\Phi(q)$.

Let d_1, \dots, d_n be the *degrees* of W , i.e. the degrees of the generators of the ring of W -invariant regular functions on \mathfrak{h} ; it is well known that $d_1 \dots d_n = |W|$. The numbers $d_1 - 1, \dots, d_n - 1$ are known as the *exponents* of W ; we denote by $\mathcal{P}(\Phi)$ their product:

$$\mathcal{P}(\Phi) \doteq (d_1 - 1) \dots (d_n - 1).$$

Then we have:

Theorem 14.

$$P_\Phi(q) = \sum_{C \in \mathcal{C}(\Phi)} \mathcal{P}(\Phi_C)(q+1)^{d(C)} q^{n-d(C)}$$

where $d(C)$ is the dimension of the layer C .

Proof. Let $nbc(\Phi)$ be the number of *no-broken circuit bases* of Φ : by [?], $nbc(\Phi)$ equals the leading coefficient of the Poincaré polynomial of the complement of \mathcal{H} in \mathfrak{h} ; moreover by [2] this coefficient is equal to $\mathcal{P}(\Phi)$ (these facts can be found also in [10, 10.1]).

Then the claim is a restatement of a known result. Indeed the cohomology of \mathcal{R}_Φ can be expressed as a direct sum of contributions given by the layers of \mathcal{T} (see for example [8, Theor. 4.2] or [10, 14.1.5]). In terms of Poincaré polynomial this expression is:

$$P_\Phi(q) = \sum_{C \in \mathcal{C}(\Phi)} nbc(\Phi_C)(q+1)^{d(C)} q^{n-d(C)}.$$

□

Now we use the theorem above to compute the Euler characteristic of \mathcal{R}_Φ .

Lemma 15.

$$E_\Phi = (-1)^n \sum_{p=0}^n \frac{|W|}{|W_p|} \mathcal{P}(\Phi_p)$$

Proof. We have

$$E_\Phi = P_\Phi(-1) = (-1)^n \sum_{t \in \mathcal{C}_0(\Phi)} \mathcal{P}(\Phi(t)) \quad (6)$$

because the contributions of all positive-dimensional layers vanish at -1 . Obviously isomorphic subsystems have the same degrees, thus Theorem 1 yields the statement. □

Theorem 16.

$$E_\Phi = (-1)^n |W|$$

Proof. By the previous lemma we must prove that

$$\sum_{p=0}^n \frac{\mathcal{P}(\Phi_p)}{|W_p|} = 1$$

If we write d_1^p, \dots, d_n^p for the degrees of W_p , the previous identity becomes

$$\sum_{p=0}^n \frac{(d_1^p - 1) \dots (d_n^p - 1)}{d_1^p \dots d_n^p} = 1.$$

This identity has been proved in [9], and later with different methods in [12]. \square

Notice that W acts on \mathcal{R}_Φ and then on its cohomology. Then we can consider the *equivariant Euler characteristic* of \mathcal{R}_Φ , that is, for each $w \in W$,

$$\tilde{E}_\Phi(w) \doteq \sum_{i=0}^n (-1)^i \text{Tr}(w, H^i(\mathcal{R}_\Phi, \mathbb{C})).$$

Let ϱ_W be the character of the regular representation of W . From Theorem 16 we get the following

Corollary 17.

$$\tilde{E}_\Phi = (-1)^n \varrho_W$$

Proof. Since W is finite and acts freely on \mathcal{R}_Φ , it is well known that $\tilde{E}_\Phi = k \varrho_W$ for some $k \in \mathbb{Z}$. Then to compute k we just have to look at $\tilde{E}_\Phi(1_W) = E_\Phi$. \square

Finally we give a formula for $P_\Phi(q)$ which, together with the mentioned results in [23], allows its explicit computation.

Theorem 18.

$$P_\Phi(q) = \sum_{d=0}^n (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |W^\Theta|$$

Proof. By formula (5) we can restate Theorem 14 as

$$P_\Phi(q) = \sum_{d=0}^n (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} \sum_{C \in \mathcal{C}_\Theta^\Phi} \mathcal{P}(\Phi_C)$$

Moreover by Theorem 11 and Corollary 12 we get

$$\sum_{C \in \mathcal{C}_\Theta^\Phi} \mathcal{P}(\Phi_C) = n_\Theta^{-1} \sum_{t \in \mathcal{C}_0(\Theta)} \mathcal{P}(\Theta(t)).$$

Finally the claim follows by formula (9) and Theorem 16 applied to Θ :

$$\sum_{t \in \mathcal{C}_0(\Theta)} \mathcal{P}(\Theta(t)) = (-1)^d \chi_\Theta = |W^\Theta|.$$

□

4.2 Examples

Case F_4 . In Section 3.3 we have given a list of all possible types of complete subsystems, together with their multiplicities. Then we just have to compute the coefficient $n_\Theta^{-1}|W^\Theta|$ for each type. This is equal to:

- 1 for types 1., 2. and 3.
- 2 for types 4. and 8.
- 4 for type 5.
- 24 for types 6. and 7.
- 1152 for type 9.

Thus

$$P_\Phi(q) = 2153q^4 + 1260q^3 + 286q^2 + 28q + 1.$$

Case A_{n-1} . By Section 1.3.3, $n_\Theta^{-1} = \frac{g_\lambda}{\lambda_1 \dots \lambda_k}$ and $|W^\Theta| = \lambda_1! \dots \lambda_k!$. Hence by Theorem 17

$$P_\Phi(q) = \sum_{d=0}^n (q+1)^d q^{n-d} \sum_{|\lambda|=d+1} n! b_\lambda^{-1} g_\lambda (\lambda_1 - 1)! \dots (\lambda_k - 1)!.$$

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